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FIXED POINT THEOREMS FOR EXTENDED GENERALIZED $\alpha-\psi-$GERAGHTY CONTRACTION TYPE MAPS IN METRIC SPACE

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Abstract: In this paper, we introduce the notion of extended generalized $\alpha-\psi-$Geraghty contraction type maps in the context of metric space and establish some fixed point theorems for such maps. Our results extend the fixed point results of Popescu [Fixed Point Theory and Applications 2014, 2014:190] in complete metric space. An example is also given to illustrate our result.

Keywords and Phrases: Metric space, fixed point, triangular $\alpha-$orbital admissible mapping, generalized $\alpha-$Geraghty contraction type map, extended generalized $\alpha-\psi-$Geraghty contraction type map.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The Banach contraction principle is one of the most fundamental results in fixed point theory. Besides being the foundation of the metric branch of fixed point theory, it is one of the most widely used fixed point theorems in all analysis. Due to its usefulness in nonlinear analysis and applications in many disciplines such as Chemistry, Physics, Biology, Computer Science, Economics, Game Theory and many branches of Mathematics, several authors have improved, generalized and extended this basic result of Banach by defining new contractive conditions and replacing the metric space by more general abstract spaces. Among such results, the works of Geraghty [6], Amini-Harandi and Emami [2], Caballero et al.[4], Gordji et al.[7], Samet et al.[17] and Karapinar and Samet [10] may be mentioned. Recently, in the line of these developments, Cho et al. [5] defined the concept of $\alpha-$Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps. Further, Erdal Karapinar [11] introduced the concept of $\alpha-\psi-$Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al.[5]. Very recently, Popescu [16] also generalized the results of Cho et al.[5] and gave other conditions to prove the existence and uniqueness of a fixed point of $\alpha-$Geraghty contraction type maps.

In this paper, motivated by the results of Popescu [16], we define extended generalized $\alpha-\psi-$Geraghty contraction type maps in the setting of metric space and obtain the existence and uniqueness of a fixed point of such maps. Our results extend the fixed point results of Popescu [16]. We also give an example to illustrate our result.

2. Preliminaries

In this section, we recall some basic definitions and related results on the topic in the literature.
Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition $\lim\limits_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$. Geraghty used such functions to prove the following result.

Theorem 2.1. [6] *Let (X,d) be a complete metric space and let T be a mapping on X. Suppose there exists $\beta \in \mathcal{F}$ such that for all $x,y \in X$, $d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$. Then T has a unique fixed point $x_* \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.*

Definition 2.2. [16] *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be $\alpha-$orbital admissible if $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx,T^2x) \geq 1$.*

Definition 2.3. [16] *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be triangular $\alpha-$orbital admissible if T is $\alpha-$orbital admissible and $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1$.*

Lemma 2.4. [16] *Let $T : X \to X$ be a triangular $\alpha-$orbital admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

Definition 2.5. [16] *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called a generalized $\alpha-$Geraghty contraction type map* *if there exists* $\beta \in \mathcal{F}$ *such that for all* $x, y \in X$,

$$\alpha(x,y)d(Tx,Ty) \leq \beta(M_T(x,y))M_T(x,y),$$

where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/2\}$.
Popescu proved the following interesting results i.e. Theorem 2.6., Theorem 2.7. and Theorem 2.8.

Theorem 2.6. [16] *Let* (X,d) *be a complete metric space,* $\alpha : X \times X \to \mathbb{R}$ *be a function and let* $T : X \to X$ *be a map. Suppose that the following conditions are satisfied:*

(1) T *is a generalized* $\alpha-$*Geraghty contraction type mapping,*

(2) T *is a triangular* $\alpha-$*orbital admissible mapping,*

(3) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$,

(4) T *is continuous.*

Then T *has a fixed point* $x^* \in X$, *and* $\{T^n x_1\}$ *converges to* x^*.

Theorem 2.7. [16] *Let* (X,d) *be a complete metric space,* $\alpha : X \times X \to \mathbb{R}$ *be a function and let* $T : X \to X$ *be a map. Suppose that the following conditions are satisfied:*

(1) T *is a generalized* $\alpha-$*Geraghty contraction type mapping,*

(2) T *is a triangular* $\alpha-$*orbital admissible mapping,*

(3) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$,

(4) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then T *has a fixed point* $x^* \in X$, *and* $\{T^n x_1\}$ *converges to* x^*.

For the uniqueness of a fixed point of a generalized $\alpha-$Geraghty contraction type map, Popescu [16] considered the following hypothesis.
(K) For all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq 1$ and $\alpha(y,v) \geq 1$ and $\alpha(v,Tv) \geq 1$.

Theorem 2.8. [16] *Replacing condition (3) with condition (K) in the hypotheses of Theorem 2.6. (resp. Theorem 2.7.), we obtain that* x^* *is the unique fixed point of* T.

3. Main Results

We now state and prove our main results. We recall the following class of auxiliary functions defined in the paper by Erdal Karapinar [11].
Let Ψ denote the class of functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions

(a) ψ is nondecreasing;

(b) ψ is subadditive, that is, $\psi(s+t) \leq \psi(s) + (t)$;

(c) ψ is continuous;

(d) $\psi(t)$=0 $\Leftrightarrow t = 0$.

Definition 3.1. *Let (X,d) be a metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Then a map $T : X \to X$ is called an extended generalized $\alpha-\psi-$Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \beta(\psi(M_T(x,y)))\psi(M_T(x,y)),$$

where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\}$ and $\psi \in \Psi$.

Theorem 3.2. *Let (X,d) be a metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions hold:*

(i) T is an extended generalized $\alpha - \psi-$Geraghty contraction type map,

(ii) T is triangular $\alpha-$orbital admissible,

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,

(iv) T is continuous.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. We construct a sequence of points $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By hypothesis, $\alpha(x_1, x_2) \geq 1$ and the map T is triangular $\alpha-$orbital admissible. Therefore by Lemma 2.4., we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.
Then, we have

$$\psi(d(x_{n+1}, x_{n+2})) = \psi(d(Tx_n, Tx_{n+1})) \leq \alpha(x_n, x_{n+1})\psi(d(Tx_n, Tx_{n+1}))$$

$$\leq \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1})) \text{ for all } n \in N \tag{1}$$

Here, we have

$$\begin{aligned}
M_T(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\
&\quad [d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)]/2\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2})/2\} \\
&\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]/2\} \\
&= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.
\end{aligned}$$

Let us suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$. Since $\beta(\psi(M_T(x_n, x_{n+1}))) < 1$, we have from (1)

$$\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2})),$$

which is a contradiction. Therefore, we must have

$$d(x_n, x_{n+1}) > d(x_{n+1}, x_{n+2}).$$

Thus, we have from (1)

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &\leq \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1})) \\
&\leq \beta(\psi(M_T(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})) \\
&< \psi(d(x_n, x_{n+1}))
\end{aligned}$$

so that $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.
Thus the sequence $\{d(x_n, x_{n+1})\}$ is positive and decreasing. Now, we prove that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence which is bounded from below.
Therefore there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We show that $r = 0$. We suppose on the contrary that $r > 0$. We have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \beta(\psi(M_T(x_n, x_{n+1}))) < 1.$$

Now by taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \beta(\psi(M_T(x_n, x_{n+1}))) = 1.$$

By the property of β, we have

$$\lim_{n\to\infty} \psi(M_T(x_n, x_{n+1})) = 0 \Rightarrow \lim_{n\to\infty} M_T(x_n, x_{n+1}) = 0.$$

This implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, which is a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0. \tag{2}$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, for all positive integers k, there exist $m_k > n_k > k$ with

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{3}$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$d(x_{m_k-1}, x_{n_k}) < \epsilon. \tag{4}$$

By (3) and (4), we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &< d(x_{m_k-1}, x_{m_k}) + \epsilon \end{aligned}$$

that is,

$$\epsilon \leq d(x_{m_k}, x_{n_k}) < \epsilon + d(x_{m_k-1}, x_{m_k}) \quad \text{for all} \quad k \in \mathbb{N}. \tag{5}$$

Then in view of (2) and (5), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{6}$$

Again, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{m_k-1}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{n_k-1}) + d(x_{m_k-1}, x_{n_k-1}) \\ d(x_{m_k-1}, x_{n_k-1}) &\leq d(x_{m_k-1}, x_{m_k}) + d(x_{n_k-1}, x_{n_k}) + d(x_{m_k}, x_{n_k}). \end{aligned}$$

Taking limit as $k \to \infty$ and using (2) and (6), we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{7}$$

Also, we have

$$|d(x_{n_k}, x_{m_k-1}) - d(x_{n_k}, x_{m_k})| \leq d(x_{m_k}, x_{m_k-1}).$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \epsilon.$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \epsilon.$$

By Lemma 2.4., we get $\alpha(x_{n_k-1}, x_{m_k-1}) \geq 1$. Therefore, we have

$$\begin{aligned} \psi(d(x_{m_k}, x_{n_k})) &= \psi(d(Tx_{m_k-1}, Tx_{n_k-1})) \\ &\leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(d(Tx_{n_k-1}, Tx_{m_k-1})) \\ &\leq \beta(\psi(M_T(x_{n_k-1}, x_{m_k-1})))\psi(M_T(x_{n_k-1}, x_{m_k-1})). \end{aligned}$$

Here we have

$$\begin{aligned} M_T(x_{n_k-1}, x_{m_k-1}) &= \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, Tx_{n_k-1}), d(x_{m_k-1}, Tx_{m_k-1}), \\ &\quad [d(x_{n_k-1}, Tx_{m_k-1}) + d(x_{m_k-1}, Tx_{n_k-1})]/2\} \\ &= \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}), \\ &\quad [d(x_{n_k-1}, x_{m_k}) + d(x_{m_k-1}, x_{n_k})]/2\} \end{aligned}$$

And we see that

$$\lim_{k\to\infty} M_T(x_{n_k-1}, x_{m_k-1}) = \epsilon.$$

Now we have

$$\frac{\psi(d(x_{n_k}, x_{m_k}))}{\psi(M_T(x_{n_k-1}, x_{m_k-1}))} \leq \beta(\psi(M_T(x_{n_k-1}, x_{m_k-1}))) < 1.$$

By using (6) and taking limit as $k \to \infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \beta(\psi(M_T(x_{n_k-1}, x_{m_k-1}))) = 1.$$

So, $\lim_{k\to\infty} \psi(M_T(x_{n_k-1}, x_{m_k-1})) = 0 \Rightarrow \lim_{k\to\infty} M_T(x_{n_k-1}, x_{m_k-1}) = 0 = \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. As T is continuous, we have $Tx_n \to Tx^*$ i.e. $\lim_{n\to\infty} x_{n+1} = Tx^*$ and so $x^* = Tx^*$. Hence x^* is a fixed point of T.

In the following Theorem, we replace the continuity of T by a suitable condition.

Theorem 3.3. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions hold:*

(i) T is an extended generalized $\alpha-\psi-$Geraghty contraction type map,

(ii) T is triangular $\alpha-$orbital admissible,

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. The proof goes along similar lines of the proof of Theorem 3.2. We conclude that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, converges to a point say $x^* \in X$. By hypothesis (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all k. Now for all k, we have

$$\begin{aligned}\psi(d(x_{n_k+1}, Tx^*)) &= \psi(d(Tx_{n_k}, Tx^*))\\ &\leq \alpha(x_{n_k}, x^*)\psi(d(Tx_{n_k}, Tx^*))\\ &\leq \beta(\psi(M_T(x_{n_k}, x^*)))\psi(M_T(x_{n_k}, x^*))\end{aligned}$$

so that

$$\psi(d(x_{n_k+1}, Tx^*)) \leq \beta(\psi(M_T(x_{n_k}, x^*)))\psi(M_T(x_{n_k}, x^*)) \tag{8}$$

On the other hand, we have

$$\begin{aligned}M_T(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*),\\ &\quad [d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})]/2\}\\ &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*),\\ &\quad [d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})]/2\}\end{aligned}$$

We suppose that $x^* \neq Tx^*$ so that $d(x^*, Tx^*) > 0$. Taking limit $k \to \infty$, we get

$$\lim_{k\to\infty} M_T(x_{n_k}, x^*) = d(x^*, Tx^*).$$

Now we have

$$\frac{\psi(d(x_{n_k+1}, Tx^*))}{\psi(M_T(x_{n_k}, x^*))} \leq \beta(\psi(M_T(x_{n_k}, x^*))) < 1.$$

And taking limit $k \to \infty$, we get $\lim\limits_{k\to\infty} \beta(\psi(M_T(x_{n_k}, x^*))) = 1$.
So, we have $\lim\limits_{k\to\infty} \psi(M_T(x_{n_k}, x^*)) = 0$ which implies that $\lim\limits_{k\to\infty} M_T(x_{n_k}, x^*) = 0$ i.e. $d(x^*, Tx^*) = 0$. This is a contradiction. Therefore we must have $x^* = Tx^*$.

For the uniqueness of a fixed point of an extended generalized $\alpha-\psi-$Geraghty contraction type map, we consider the following hypothesis:
(K1) For all $x \neq y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$, $\alpha(y,z) \geq 1$ and $\alpha(z,Tz) \geq 1$.

Theorem 3.4. *Adding condition (K1) to the hypotheses of Theorem 3.2.(or Theorem 3.3.), we obtain that x^* is the unique fixed point of T.*
Proof. Due to Theorem 3.2. (or Theorem 3.3.), we obtain that $x^* \in X$ is a fixed point of T. Let $y^* \in X$ be another fixed point of T such that $x^* \neq y^*$. Then by hypothesis (K1), there exists $z \in X$ such that $\alpha(x^*,z) \geq 1, \alpha(y^*,z) \geq 1$ and $\alpha(z,Tz) \geq 1$.
Since T is $\alpha-$orbital admissible, we get $\alpha(x^*,T^nz) \geq 1$ and $\alpha(y^*,T^nz) \geq 1$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \psi(d(x^*,T^{n+1}z)) &\leq \alpha(x^*,T^nz)\psi(d(Tx^*,TT^nz)) \\ &\leq \beta(\psi(M_T(x^*,T^nz)))\psi(M_T(x^*,T^nz)), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Here we have

$$\begin{aligned} M_T(x^*,T^nz) &= \max\{d(x^*,T^nz), d(x^*,Tx^*), d(T^nz,TT^nz), \\ &\quad [d(x^*,TT^nz)+d(T^nz,Tx^*)]/2\} \\ &= \max\{d(x^*,T^nz), d(T^nz,T^{n+1}z), [d(x^*,T^{n+1}z)+d(x^*,T^nz)]/2\} \end{aligned}$$

By Theorem 3.2. (or Theorem 3.3.) we deduce that the sequence $\{T^nz\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$ we get $\lim_{n\to\infty} M_T(x^*,T^nz) = d(x^*,z^*)$. Let us suppose that $z^* \neq x^*$. Then we have

$$\frac{\psi(d(x^*,T^{n+1}z))}{\psi(M_T(x^*,T^nz))} \leq \beta(\psi(M_T(x^*,T^nz))) < 1.$$

Taking limit $n \to \infty$, we get $\lim_{n\to\infty} \beta(\psi(M_T(x^*,T^nz))) = 1$.
Therefore we have $\lim_{n\to\infty} \psi(M_T(x^*,T^nz)) = 0$. This implies that $\lim_{n\to\infty} M_T(x^*,T^nz) = 0$ i.e. $d(x^*,z^*) = 0$, which is a contradiction. Therefore we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus we have $y^* = x^*$. Hence x^* is the unique fixed point of T.
Remark. If we take $\psi(t) = t$ in Theorem 3.2., Theorem 3.3. and Theorem 3.4., we respectively get Theorem 2.6., Theorem 2.7. and Theorem 2.8. Now, we give an example to illustrate Theorem 3.3.

Example 3.5. Let $X = [-2,-1] \cup \{0\} \cup [1,2]$ and let $d(x,y) = |x-y|$ for all

$x, y \in X$. Then (X, d) is a complete metric space. And let $\beta(t) = \frac{1}{2}$ for all $t \geq 0$. Then $\beta \in \mathcal{F}$. Also let the function $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = \frac{t}{2}$. Then we have $\psi \in \Psi$. Let a map $T : X \to X$ be defined by

$$Tx = \begin{cases} -x & \text{if} \quad x \in [-2, -1) \cup (1, 2], \\ 0 & \text{if} \quad x \in \{-1, 0, 1\}. \end{cases}$$

And let a function $\alpha : X \times X \to \mathbb{R}$ be defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad xy \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

If $\alpha(x, Tx) \geq 1$, then $xTx \geq 0$. This implies that $Tx = 0$ and so $\alpha(Tx, T^2x) \geq 1$. Also if $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$, then $Ty = 0$. Thus $xTy = 0$ and so $\alpha(x, Ty) \geq 1$. Therefore, T is triangular $\alpha-$orbital admissible. Condition (iii) of Theorem 3.3. is satisfied with $x_1 = 1$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $x_n x \geq 0$ and so $\alpha(x_n, x) \geq 1$ for all n.

We finally show that condition (i) of Theorem 3.3. is satisfied. If $x, y \in [-2, -1)$, then $\alpha(x, y) = 1, d(Tx, Ty) = |x - y| \leq 1$ and $M_T(x, y) \geq -2x \geq 2$. Therefore

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \frac{1}{2} \quad \text{and} \quad \beta(\psi(M_T(x, y)))\psi(M_T(x, y)) \geq \frac{1}{2}.$$

Thus we have

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)).$$

The case $x, y \in (1, 2]$ is similar.

If $x \in [-2, -1) \cup (1, 2]$ and $y \in \{-1, 0, 1\}$, then $d(Tx, Ty) = |x|$, $M_T(x, y) \geq 2|x|$, so $d(Tx, Ty) \leq M_T(x, y)/2$. Also, we have $\alpha(x, y) = 0$ or 1. Thus, we have

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)).$$

If $x, y \in \{-1, 0, 1\}$, then $d(Tx, Ty) = 0 \leq M_T(x, y)/2$. And, $\alpha(x, y) = 0$ or 1. Therefore, we have

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)).$$

Further, if $x \in [-2, -1), y \in (1, 2]$, then $\alpha(x, y) = 0$. Therefore, we have

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)).$$

Thus all the conditions of Theorem 3.3. are satisfied and T has a unique fixed point $x^* = 0$.

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